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Comparison of CSC method and the B-net method for deducing smoothness condition

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Abstract

The first author of this paper established an approach to study the multivariate spline over arbitrary partition, and presented the so-called conformality method of smoothing cofactor (the CSC method). Farin introduced the B-net method which is suitable for studying the multivariate spline over simplex partitions. This paper indicates that the smoothness conditions obtained in terms of the B-net method can be derived by the CSC method for the spline spaces over simplex partitions, and the CSC method is more capable in some sense than the B-net method in studying the multivariate spline.

Keywords: Multivariate spline; Smoothing cofactor; Global conformality condition; B-net method; Smoothness condition

1. Introduction

Splines are piecewise polynomials with certain smoothness. The first author of this paper established the basic theory on multivariate spline over arbitrary partition, and presented the so-called conformality method of smoothing cofactor (the CSC method) which is suitable for studying the multivariate spline over arbitrary partition [1].

In this paper we take the bivariate spline as an example to prove that the CSC method and the B-net method are equivalent over simplex partitions. The CSC method and the B-net method on bivariate spline spaces are presented in Section 2. In Section 3, we derive the smoothness conditions over triangulation with the CSC method, which are the same as the smoothness conditions presented by Farin [2,3]. Finally, we indicate that the CSC method and the B-net method are equivalent for multivariate spline spaces over simplex partitions.

2. Bivariate spline spaces

Let D be a domain in R^2, P_k the collection of all these bivariate polynomials with real coefficients and total degree no more than k, i.e.,

$$P_k := \left\{ p = \sum_{i=0}^{k} \sum_{j=0}^{k-i} c_{ij} x^i y^j | c_{ij} \in R \right\}$$

Using a finite number of irreducible algebraic curves to carry out the partition Δ of the domain D, then the domain D is divided into N sub-domains $\delta_1, \ldots, \delta_N$, each of such sub-domains is called a cell of Δ. These line segments that form the boundary of each cell are called the edges, intersection points of the edges are called the vertices. If two vertices are two end points of a single edge, then these two vertices are called the adjacent vertices. The vertices which are not lying on the boundary of domain D are called interior

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vertices. The space of bivariate spline with degree k and smoothness μ over Δ is defined by

$$S_k^\mu(\Delta) := \{s \in C^\mu(D) | s|_{\delta_i} \in P_k,\ i = 1, \ldots, N\}$$

2.1. The conformality method of smoothing cofactor

Theorem 1. [1]. *Let the representation of $z = s(x, y)$ on the two arbitrary adjacent cells D_i, and D_j be*

$$z = p_i(x, y), \quad \text{and} \quad z = p_j(x, y)$$

where $z = p_i(x, y)$, and $z = p_j(x, y) \in P_k$, respectively. In order to let $s(x, y) \in C^\mu(\overline{D_i \bigcup D_j})$, if and only if there is a polynomial $q_{ij}(x, y) \in P_{k-(\mu+1)d}$, such that

$$p_i(x, y) - p_j(x, y) = [l_{ij}(x, y)]^{\mu+1} \cdot q_{ij}(x, y) \tag{1}$$

where $\overline{D}_i$, and $\overline{D}_j$ have the common interior edge

$$\Gamma_{ij} : l_{ij}(x, y) = 0$$

and the irreducible algebraic polynomial $l_{ij}(x, y) \in P_d$.

The polynomial $q_{ij}(x, y)$ defined by Eq. (1) in Theorem 1 is called the smoothing cofactor of $s(x, y)$ across Γ_{ij} from D_j to D_i.

Let A be a given interior vertex over partition Δ, the conformality condition at A is defined by

$$\sum_A [l_{ij}(x, y)]^{\mu+1} \cdot q_{ij}(x, y) \equiv 0$$

where $\sum_A$ presents the summation of all the interior edges around A, and $q_{ij}(x, y)$ is the smoothing cofactor across Γ_{ij}.

Let $A_1, \ldots, A_M$ be all the interior vertices over partition Δ. The global conformality condition is defined by

$$\sum_{A_v} [l_{ij}(x, y)]^{\mu+1} \cdot q_{ij}(x, y) \equiv 0, \quad v = 1, \ldots, M \tag{2}$$

Theorem 2. [1]. *Let Δ be any partition of D. The bivariate spline function $s(x, y) \in S_k^\mu(\triangle)$ exists, if and only if for every interior edge, there exists a smoothing cofactor of $s(x, y)$, and the global conformality condition Eq. (2) is satisfied.*

Definition 1. [1]. The partition Δ is called a cross-cut partition, if all the edges are lying on some straight lines cross-cutting domain D. We call a partition to be quasi-cross-cut denoted by Δ_{qc}, if each edge in this partition is either a part of cross-cut or a part of rays in D.

Definition 2. [1]. The union of all the cells sharing the same interior vertex V is called the relative region (or star-region) of the interior vertex V.

Let V_N be the solution space corresponding to the conformality condition at an interior vertex, where N is the number of lines passing though this interior vertex, and having different slopes. The dimension of V_N is presented as follows.

Lemma 1. [4].

$$d_k^\mu(N) = \frac{1}{2}\left(k - \mu - \left[\frac{\mu+1}{N-1}\right]\right)_+ \cdot \left((N-1)k - (N+1)\mu + (N-3) + (N-1)\left[\frac{\mu+1}{N-1}\right]\right) \tag{3}$$

Theorem 3. [4]. *Let Δ_{qc} be a quasi-cross-cut partition of a simply connected region, Δ_{qc} have L_1 cross-cuts, L_2 rays, and V interior vertices $A_1, \ldots, A_V$. Denote by N_i, $i = 1, \ldots, V$ the number of cross-cuts, and rays passing through A_i. We have*

$$dim S_k^\mu(\Delta_{qc}) = \binom{k+2}{2} + L_1 \binom{k-\mu+1}{2} + \sum_{i=1}^{V} d_k^\mu(N_i) \tag{4}$$

where $d_k^\mu(N)$ is given in Eq. (3).

2.2. The B-net method

The B-net method is suitable for studying the spline functions over arbitrary simplex partition. Now we introduce the main idea of the B-net method of bivariate spline spaces over simplices [3].

It is well known that any point x in the plane can be uniquely expressed in terms of barycentric coordinates with respect to any nondegenerate triangle $\triangle$ with vertices v_1, v_2, v_3 (see Fig. 1, left):

$$x = \tau_1 v_1 + \tau_2 v_2 + \tau_3 v_3$$

where $\tau := (\tau_1, \tau_2, \tau_3)$ is usually normalized by the requirement

$$\tau_1 + \tau_2 + \tau_3 = 1$$

and the coefficients $\tau := (\tau_1, \tau_2, \tau_3)$ are called the barycentric coordinates of x over the triangle $\triangle$.

We have

$$\tau_1 = \frac{det(v_2 - x, v_3 - x)}{det(v_2 - v_1, v_3 - v_1)}, \quad \tau_2 = \frac{det(v_1 - x, v_3 - x)}{det(v_1 - v_2, v_3 - v_2)}, \quad \tau_3 = \frac{det(v_1 - x, v_2 - x)}{det(v_1 - v_3, v_2 - v_3)}$$

An important property of barycentric coordinates is affine invariance.

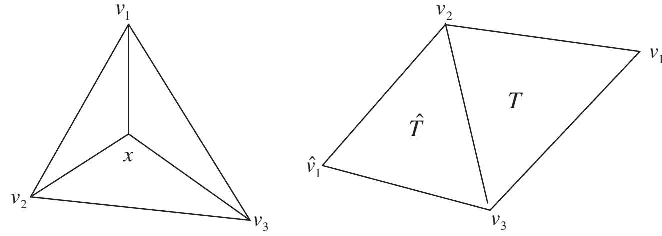

Fig. 1. Triangle $\triangle$ (left) and two adjacent triangles, T and $\widehat{T}$ (right).

Let
$\lambda := (\lambda_1, \lambda_2, \lambda_3),\ |\lambda| = \lambda_1 + \lambda_2 + \lambda_3 = n,\ \lambda! = \lambda_1!\lambda_2!\lambda_3!$.
Bernstein polynomials of degree n over a triangle are defined by

$$B^n_\lambda(\tau) = \frac{n!}{\lambda!}\tau^\lambda = \frac{n!}{\lambda_1!\lambda_2!\lambda_3!}\tau_1^{\lambda_1}\tau_2^{\lambda_2}\tau_3^{\lambda_3}, \quad \lambda_1 + \lambda_2 + \lambda_3 = n,$$
$$\lambda_i \in Z_+,\ i = 1, 2, 3$$

There are many properties of Bernstein polynomials [5], such as

(1) $B^n_\lambda(\tau) \geqslant 0$, if $\tau \in \triangle = [v_1, v_2, v_3]$.
(2) $\sum_{|\lambda|=n} B^n_\lambda(\tau) \equiv 1$.
(3) $\{B^n_\lambda(\tau),\ |\lambda| = n\}$ is a basis of the polynomial space P_n.
(4) $B^n_\lambda(\tau)$ has a unique maximum value at point $\tau = \frac{\lambda}{n}$.

From property (3), we have

Lemma 2. [5]. *Any polynomial $P \in P_n$ can be uniquely expressed as*

$$P(\tau) = \sum_{|\lambda|=n} b_\lambda B^n_\lambda(\tau) \tag{5}$$

where $\{b_\lambda,\ |\lambda| = n\}$ are called the Bézier coordinates of $P(\tau)$ over $\triangle$, the piecewise linear function interpolating to $\{(\frac{\lambda}{n}, b_\lambda) : |\lambda| = n\}$ is called the Bézier net of $P(\tau)$ over $\triangle$, B-net for shot.

Let $v_1,\ v_2,\ v_3$ be the vertices of triangle T, and $\hat{v}_1,\ v_2,\ v_3$ be the vertices of triangle $\widehat{T}$. T and $\widehat{T}$ have the common boundary v_2v_3 (see Fig. 1, right). The smoothness conditions of polynomials of degree n over two adjacent triangles are presented as follows.

Theorem 4. [3]. *Let $P(\tau)$ and $\widehat{P}(\tau)$ denote polynomials of degree n defined on $T = [v_1, v_2, v_3]$, and $\widehat{T} = [\hat{v}_1, v_2, v_3]$, respectively. Let $\{b_\lambda,\ |\lambda| = n\}$ and $\{\widehat{b}_\lambda,\ |\lambda| = n\}$ be the Bézier coordinates of $P(\tau)$ over T and $\widehat{P}(\tau)$ over $\widehat{T}$, respectively. A necessary and sufficient condition for $P(\tau)$ and $\widehat{P}(\tau)$ to be C^r across the common boundary is*

$$\hat{b}_{\lambda^t} = b^t_{\lambda^0}(\sigma), \quad t = 0, 1, \cdots, r \tag{6}$$

where

$$b^r_\lambda(\sigma) = \sum_{|\mu|=r} b_{\lambda+\mu} B^r_\mu(\sigma); \quad |\lambda| = n - r \tag{7}$$

σ is the barycentric coordinate of $\hat{v}_1$ over T, $\lambda^t = (t, \lambda_2, \lambda_3),\ \lambda^0 = (0, \lambda_2, \lambda_3),\ \lambda_2 + \lambda_3 = n - t$.

Definition 3. [6]. Let Δ denote the simplex partition on domain D, and let Γ denote the set of control points of a spline in $S^\mu_k(\triangle)$. A subset $\Lambda \subseteq \Gamma$ is a determining set for $S^\mu_k(\triangle)$ if

$$s(x) = 0,\ \forall x \in \Lambda \Rightarrow s(x) = 0,\ \forall x \in \Gamma$$

Λ is a minimal determining set if there is no smaller determining set.

3. Deriving the B-net method with the conformality method of smoothing cofactor

By the definition of the barycentric coordinates, we have

Lemma 3. *Let $b^k(\hat{\tau})$, and $c^k(\tau)$ denote polynomials of degree k defining over two adjacent triangles $\widehat{T} = [\hat{v}_1, v_2, v_3]$ and $T = [v_1, v_2, v_3]$, respectively. Denote by $\sigma(\sigma_1, \sigma_2, \sigma_3)$ the barycentric coordinates of v_1 over $\widehat{T}$. The relations between the barycentric coordinates over the adjacent triangles are as follows*

$$\hat{\tau}_1 = \sigma_1 \cdot \tau_1, \quad \hat{\tau}_2 = \sigma_2 \cdot \tau_1 + \tau_2, \quad \hat{\tau}_3 = \sigma_3 \cdot \tau_1 + \tau_3 \tag{8}$$

3.1. $S^1_3(\Delta)$ over two adjacent triangles

Denote by $b^3(\hat{\tau})$ and $c^3(\tau)$ the bivariate polynomials of degree 3 defining over two adjacent triangles $\widehat{T} = [\hat{v}_1, v_2, v_3]$ and $T = [v_1, v_2, v_3]$, respectively (see Fig. 2).

Let $\{b_\eta : |\eta| = 3\}$ and $\{c_\lambda : |\lambda| = 3\}$ be the Bézier coordinates of $b^3(\hat{\tau})$ over $\widehat{T}$ and $c^3(\tau)$ over T, respectively. Denote by $\sigma(\sigma_1, \sigma_2, \sigma_3)$ the barycentric coordinates of v_1 over $\widehat{T}$. The expression of $b^3(\hat{\tau})$ is

$$\begin{aligned} b^3(\hat{\tau}) = \sum_{|\eta|=3} b_\eta B^3_\eta(\hat{\tau}) &= b_{3,0,0}\hat{\tau}_1^3 + 3b_{2,1,0}\hat{\tau}_1^2\hat{\tau}_2 + 3b_{1,2,0}\hat{\tau}_1\hat{\tau}_2^2 \\ &+ b_{0,3,0}\hat{\tau}_2^3 + 3b_{0,2,1}\hat{\tau}_2^2\hat{\tau}_3 + 3b_{0,1,2}\hat{\tau}_2\hat{\tau}_3^2 + b_{0,0,3}\hat{\tau}_3^3 \\ &+ 3b_{1,0,2}\hat{\tau}_1\hat{\tau}_3^2 + 3b_{2,0,1}\hat{\tau}_1^2\hat{\tau}_3 + 6b_{1,1,1}\hat{\tau}_1\hat{\tau}_2\hat{\tau}_3 \end{aligned}$$

By Lemma 3, we have

$$\hat{\tau}_1 = \sigma_1 \cdot \tau_1, \quad \hat{\tau}_2 = \sigma_2 \cdot \tau_1 + \tau_2, \quad \hat{\tau}_3 = \sigma_3 \cdot \tau_1 + \tau_3$$

Denote

$$\begin{aligned} m_1 &= \sigma_1^3 b_{3,0,0} + 3\sigma_1^2\sigma_2 b_{2,1,0} + 3\sigma_1\sigma_2^2 b_{1,2,0} + \sigma_2^3 b_{0,3,0} \\ &+ 3\sigma_2^2\sigma_3 b_{0,2,1} + 3\sigma_2\sigma_3^2 b_{0,1,2} + \sigma_3^3 b_{0,0,3} + 3\sigma_1\sigma_3^2 b_{1,0,2} \\ &+ 3\sigma_1^2\sigma_3 b_{2,0,1} + 6\sigma_1\sigma_2\sigma_3 b_{1,1,1} \\ m_2 &= \sigma_1^2 b_{2,1,0} + 2\sigma_1\sigma_2 b_{1,2,0} + \sigma_2^2 b_{0,3,0} + 2\sigma_2\sigma_3 b_{0,2,1} + \sigma_3^2 b_{0,1,2} \\ &+ 2\sigma_1\sigma_3 b_{1,1,1} \\ m_3 &= \sigma_1^2 b_{2,0,1} + 2\sigma_1\sigma_2 b_{1,1,1} + \sigma_2^2 b_{0,2,1} + 2\sigma_2\sigma_3 b_{0,1,2} + \sigma_3^2 b_{0,0,3} \\ &+ 2\sigma_1\sigma_3 b_{1,0,2} \end{aligned}$$

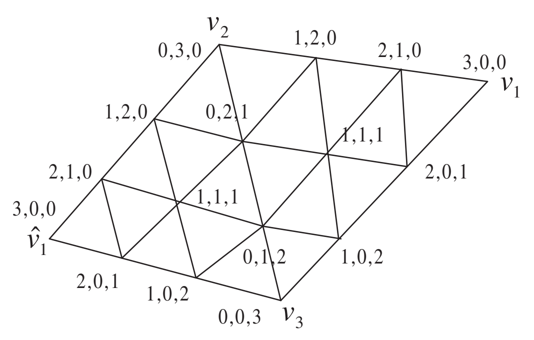

Fig. 2. $S^1_3(\triangle)$.

From Eq. (8),

$$b^3(\hat{\tau}) = m_1\tau_1^3 + 3m_2\tau_1^2\tau_2 + 3m_3\tau_1^2\tau_3 + 3(\sigma_1 b_{1,2,0} + \sigma_2 b_{0,3,0} + \sigma_3 b_{0,2,1})\tau_1\tau_2^2 + 3(\sigma_1 b_{1,0,2} + \sigma_2 b_{0,1,2} + \sigma_3 b_{0,0,3})\tau_1\tau_3^2 + 6(\sigma_1 b_{1,1,1} + \sigma_2 b_{0,2,1} + \sigma_3 b_{0,1,2})\tau_1\tau_2\tau_3 + b_{0,3,0}\tau_2^3 + 3b_{0,2,1}\tau_2^2\tau_3 + 3b_{0,1,2}\tau_2\tau_3^2 + b_{0,0,3}\tau_3^3$$

The expression of $c^3(\tau)$ is

$$c^3(\tau) = \sum_{|\lambda|=3} c_\lambda B_\lambda^3(\tau) = c_{3,0,0}\tau_1^3 + 3c_{2,1,0}\tau_1^2\tau_2 + 3c_{1,2,0}\tau_1\tau_2^2 + c_{0,3,0}\tau_2^3 + 3c_{0,2,1}\tau_2^2\tau_3 + 3c_{0,1,2}\tau_2\tau_3^2 + c_{0,0,3}\tau_3^3 + 3c_{1,0,2}\tau_1\tau_3^2 + 3c_{2,0,1}\tau_1^2\tau_3 + 6c_{1,1,1}\tau_1\tau_2\tau_3$$

Notice that the expression of the common boundary v_2v_3 is $\tau_1 = 0$. Let $b^3(\hat{\tau})$, and $c^3(\tau)$ be C^1 across the common boundary. By Theorem 1, there is a polynomial $q(\tau)$ of degree 1, such that

$$c^3(\tau) - b^3(\hat{\tau}) = q(\tau)\tau_1^2$$

So

$$c_{0,3,0} = b_{0,3,0},\ c_{0,2,1} = b_{0,2,1},\ c_{0,1,2} = b_{0,1,2},\ c_{0,0,3} = b_{0,0,3}$$
$$c_{1,2,0} = \sigma_1 b_{1,2,0} + \sigma_2 b_{0,3,0} + \sigma_3 b_{0,2,1}$$
$$c_{1,0,2} = \sigma_1 b_{1,0,2} + \sigma_2 b_{0,1,2} + \sigma_3 b_{0,0,3}$$
$$c_{1,1,1} = \sigma_1 b_{1,1,1} + \sigma_2 b_{0,2,1} + \sigma_3 b_{0,1,2}$$

It indicates that the necessary and sufficient conditions for polynomials of degree 3 defining over two adjacent triangles to be C^1 across the common boundary are that the Bézier coordinates of the two polynomials satisfy the relations above. This is the same as Theorem 4.

Moreover, we obtain the expression of the smoothing cofactor across the common boundary v_2v_3

$$q(\tau) = (c_{3,0,0} - m_1)\tau_1^3 + 3(c_{2,1,0} - m_2)\tau_1^2\tau_2 + 3(c_{2,0,1} - m_3)\tau_1^2\tau_3$$

Next, we derive the smoothness conditions obtained from the B-net method with the conformality method of smoothing cofactor.

3.2. $S_k^\mu(\Delta)$ over two adjacent triangles

Theorem 5. *Let $b^k(\hat{\tau})$ and $c^k(\tau)$ denote polynomials of degree k defining over two adjacent triangles $\hat{T} = [\hat{v}_1, v_2, v_3]$ and $T = [v_1, v_2, v_3]$, respectively. Let $\{b_\eta : |\eta| = k\}$ and $\{c_\lambda : |\lambda| = k\}$ be the Bézier coordinates of $b^k(\hat{\tau})$ over $\hat{T}$ and $c^k(\tau)$ over T, respectively. Denote by $\sigma(\sigma_1, \sigma_2, \sigma_3)$ the barycentric coordinates of v_1 over $\hat{T}$. Let $\triangle = \hat{T} \bigcup T$, $s(x,y) \in S_k^\mu(\triangle)$, $p_1(x,y)$, and $p_2(x,y)$ be the expressions of $s(x,y)$ over $\hat{T}$ and T, respectively, where $p_1(x,y)$ and $p_2(x,y) \in P_k$. Then the following conditions are equivalent to each other.*

(i) There is a smoothing cofactor $q(x,y) \in P_{k-\mu-1}$ across the common boundary v_2v_3, such that

$$p_2(x,y) - p_1(x,y) = q(x,y) \cdot l(x,y)^{\mu+1} \tag{9}$$

where $l(x,y) = 0$ is the equation of v_2v_3.

(ii)

$$c_{\lambda^t} = b_{\lambda^0}^t(\sigma), \quad t = 0, 1, \cdots, \mu \tag{10}$$

where $\lambda^t = (t, \lambda_2, \lambda_3),\ \lambda^0 = (0, \lambda_2, \lambda_3),\ \lambda_2 + \lambda_3 = k - t.$

Proof. By Lemma 2, $b^k(\hat{\tau})$ and $c^k(\tau)$ can be expressed as

$$b^k(\hat{\tau}) = \sum_{|\eta|=k} b_\eta B_\eta^k(\hat{\tau}) = \sum_{|\eta|=k} b_\eta \frac{k!}{\eta!}\hat{\tau}^\eta \tag{11}$$

and

$$c^k(\tau) = \sum_{|\lambda|=k} c_\lambda B_\lambda^k(\tau) = \sum_{|\lambda|=k} c_\lambda \frac{k!}{\lambda!}\tau^\lambda \tag{12}$$

From Eq. (8)

$$\begin{aligned} b^k(\hat{\tau}) &= \sum_{|\eta|=k} b_\eta \frac{k!}{\eta_1!\eta_2!\eta_3!}(\sigma_1 \cdot \tau_1)^{\eta_1}(\sigma_2 \cdot \tau_1 + \tau_2)^{\eta_2}(\sigma_3 \cdot \tau_1 + \tau_3)^{\eta_3} \\ &= \sum_{|\eta|=k} b_\eta \frac{k!}{\eta_1!\eta_2!\eta_3!}\sigma_1^{\eta_1}\tau_1^{\eta_1}\sum_{i=0}^{\eta_2}\binom{\eta_2}{i}\sigma_2^i\tau_1^i\tau_2^{\eta_2-i}\sum_{j=0}^{\eta_3}\binom{\eta_3}{j}\sigma_3^j\tau_1^j\tau_3^{\eta_3-j} \\ &= \sum_{|\eta|=k} b_\eta \frac{k!}{\eta_1!\eta_2!\eta_3!}\sum_{i=0}^{\eta_2}\sum_{j=0}^{\eta_3}\binom{\eta_2}{i}\binom{\eta_3}{j}\sigma_1^{\eta_1}\sigma_2^i\sigma_3^j\tau_1^{\eta_1+i+j}\tau_2^{\eta_2-i}\tau_3^{\eta_3-j} \\ &= \sum_{|\eta|=k} b_\eta \frac{k!}{\eta_1!}\sum_{i=0}^{\eta_2}\sum_{j=0}^{\eta_3}\frac{1}{(\eta_2-i)!(\eta_3-j)!i!j!}\sigma_1^{\eta_1}\sigma_2^i\sigma_3^j\tau_1^{\eta_1+i+j}\tau_2^{\eta_2-i}\tau_3^{\eta_3-j} \end{aligned}$$

Denote

$$r := (r_1, r_2, r_3) := (\eta_1, i, j),\ |\ r\ | = \lambda_1,\ \eta_2 - i = \lambda_2,\ \eta_3 - j = \lambda_3$$

It is clear that

$$\eta_2 = \lambda_2 + i,\ \eta_3 = \lambda_3 + j,$$
$$\eta := (\eta_1, \eta_2, \eta_3) := (0, \lambda_2, \lambda_3) + (r_1, r_2, r_3)$$

So $b^k(\hat{\tau})$ can be simplified as

$$\begin{aligned} b^k(\hat{\tau}) &= \sum_{|\lambda|=k}\sum_{|r|=\lambda_1} b_\eta \frac{k!}{r!\lambda_2!\lambda_3!}\sigma^r\tau^\lambda \\ &= \sum_{|\lambda|=k}\sum_{|r|=\lambda_1} b_{(0,\lambda_2,\lambda_3)+(r_1,r_2,r_3)}\frac{k!}{r!\lambda_2!\lambda_3!}\sigma^r\tau^\lambda \end{aligned} \tag{13}$$

Comparing Eq. (12) with Eq. (13), we have

$$c^k(\tau) - b^k(\hat{\tau}) = \sum_{|\lambda|=k}\left(\frac{c_\lambda}{\lambda_1!} - \sum_{|r|=\lambda_1} b_{(0,\lambda_2,\lambda_3)+(r_1,r_2,r_3)}\frac{1}{r!}\sigma^r\right)\frac{k!}{\lambda_2!\lambda_3!}\tau^\lambda$$

Let $\lambda_1 = t$, then

$$\begin{aligned} c^k(\tau) - b^k(\hat{\tau}) &= \sum_{t=0}^{k}\sum_{|\lambda|=k}\left(c_{\lambda^t} - b_{\lambda^0}^t(\sigma)\right)\frac{k!}{t!\lambda_2!\lambda_3!}\tau_1^t\tau_2^{\lambda_2}\tau_3^{\lambda_3} \\ &= \sum_{t=0}^{\mu}\sum_{|\lambda|=k}\left(c_{\lambda^t} - b_{\lambda^0}^t(\sigma)\right)\frac{k!}{t!\lambda_2!\lambda_3!}\tau_1^t\tau_2^{\lambda_2}\tau_3^{\lambda_3} \\ &\quad + \sum_{t=\mu+1}^{k}\sum_{|\lambda|=k}\left(c_{\lambda^t} - b_{\lambda^0}^t(\sigma)\right)\frac{k!}{t!\lambda_2!\lambda_3!}\tau_1^t\tau_2^{\lambda_2}\tau_3^{\lambda_3} \end{aligned} \tag{14}$$

Deriving (ii) with (i). There is a smoothing cofactor $q(x,y) \in P_{k-\mu-1}$ across the common boundary v_2v_3, such that

$$p_2(x,y) - p_1(x,y) = q(x,y) \cdot l(x,y)^{\mu+1}$$

where $l(x,y) = 0$ is the equation of v_2v_3, and its barycentric coordinate over T is $\tau_1 = 0$. So the first part of Eq. (14) should be zero, that is

$$c_{\lambda^t} = b^t_{\lambda^0}(\sigma), \quad t = 0, 1, \cdots, \mu$$

where
$|\lambda| = k,\ \lambda^t = (t, \lambda_2, \lambda_3),\ \lambda^0 = (0, \lambda_2, \lambda_3),\ \lambda_2 + \lambda_3 = k - t.$

Deriving (i) with (ii). It is known that

$$c_{\lambda^t} = b^t_{\lambda^0}(\sigma),\ t = 0, 1, \ldots, \mu, |\lambda| = k,\ \lambda^t = (t, \lambda_2, \lambda_3),$$
$$\lambda^0 = (0, \lambda_2, \lambda_3),\ \lambda_2 + \lambda_3 = k - t$$

So the first part of Eq. (14) is zero. Moreover, there is a polynomial

$$q(\tau) = \sum_{t=\mu+1}^{k} \sum_{|\lambda|=k} (c_{\lambda^t} - b^t_{\lambda^0}(\sigma)) \frac{k!}{t!\lambda_2!\lambda_3!} \tau_1^{t-\mu-1} \tau_2^{\lambda_2} \tau_3^{\lambda_3}$$

such that

$$c^k(\tau) - b^k(\hat{\tau}) = q(\tau)\tau_1^{\mu+1}$$

Obviously, $q(\tau)$ is the smoothing cofactor across the common boundary v_2v_3.

Theorem 5 indicates that both the existence of the smoothing cofactor and the smoothness conditions obtained from the B-net method are equivalent over two adjacent triangles. □

3.3. $S^\mu_k(\Delta^*)$ on the star-region over triangulation

Let Δ^* be a triangulation shown in Fig. 3, and V_0 be the common vertex of triangles T_1, T_2, and T_3. Denote by

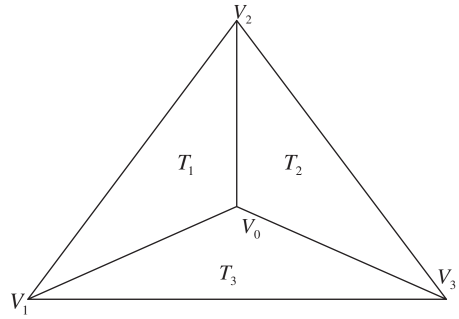

Fig. 3. Triangle $\triangle^*$.

$\sigma_1(\sigma_{11}, \sigma_{12}, \sigma_{13})$, $\sigma_2(\sigma_{21}, \sigma_{22}, \sigma_{23})$, and $\sigma_3(\sigma_{31}, \sigma_{32}, \sigma_{33})$ the barycentric coordinates of three vertexes V_3, V_1, and V_2 over T_1, T_2, and T_3, respectively. We have

Lemma 4.

$$\begin{array}{llll} \sigma_{11}\sigma_{21} = 1, & \sigma_{12}\sigma_{31} = 1, & \sigma_{23} = -1, & \sigma_{11} = \sigma_{13} \\ \sigma_{32} = \sigma_{33}, & \sigma_{12} + \sigma_{11}\sigma_{22} = 0, & \sigma_{13} + \sigma_{12}\sigma_{32} = 0 & \end{array} \quad (15)$$

Proof. Let $b^{(i)}(\tau_i)$ $(\tau_i := (\tau_{i1}, \tau_{i2}, \tau_{i3}),\ i = 1, 2, 3)$ be the polynomials of degree k defining over T_i. By Lemma 3, we have

$$\tau_{11} = \sigma_{11}\tau_{21}, \quad \tau_{21} = \tau_{31} + \sigma_{21}\tau_{32},$$
$$\tau_{31} = \sigma_{31}\tau_{12}, \quad \tau_{32} = \tau_{11} + \sigma_{32}\tau_{12}$$

So

$$\begin{aligned} \tau_{11} &= \sigma_{11}(\tau_{31} + \sigma_{21}\tau_{32}) = \sigma_{11}\tau_{31} + \sigma_{11}\sigma_{21}\tau_{32} \\ &= \sigma_{11}\sigma_{31}\tau_{12} + \sigma_{11}\sigma_{21}(\tau_{11} + \sigma_{32}\tau_{12}) \\ &= \sigma_{11}\sigma_{21}\tau_{11} + (\sigma_{11}\sigma_{31} + \sigma_{11}\sigma_{21}\sigma_{32})\tau_{12} \end{aligned}$$

Obviously, $\sigma_{11}\sigma_{21} = 1$. Others can be proved similarly. □

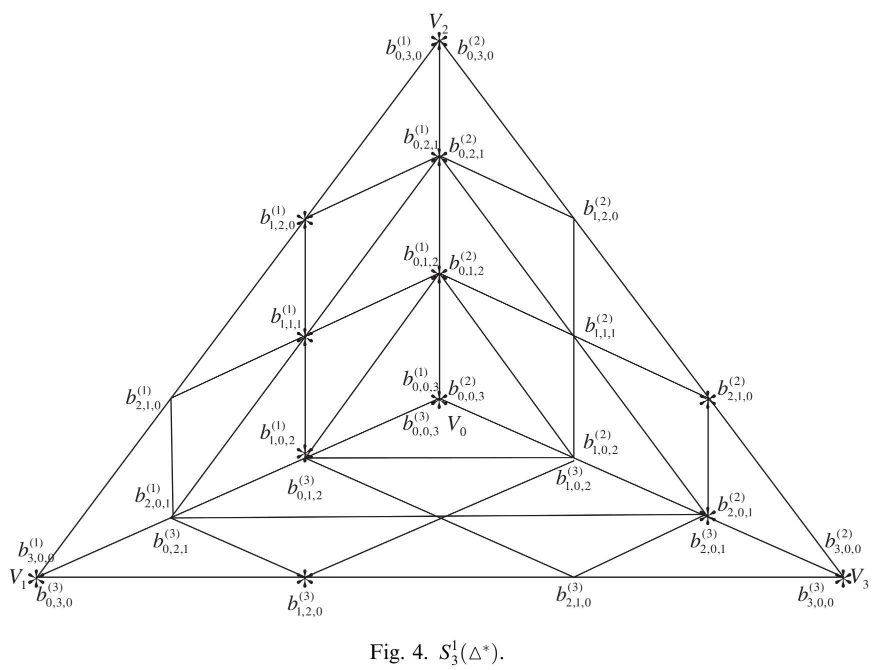

Fig. 4. $S^1_3(\triangle^*)$.

We can get some conditions of the Bézier coordinates between two adjacent simplexes which satisfy certain smoothness. Then we find all the conditions of the Bézier coordinates over the whole partition. Taking $S_3^1(\Delta^*)$ for example (see Fig. 4), Λ is one of minimal determining sets [6] for $S_3^1(\Delta^*)$, where $|\Lambda| = 12$, we mark all the control points belonging to Λ with $*$. We also have $dim S_3^1(\Delta^*) = 12$ by Theorem 3.

Theorem 6. *Suppose that V_0 is the common interior vertex of triangles T_1, T_2, and T_3 in the partition Δ^*. Let $b^{(i)}(\tau_i)$ denote polynomials of degree k defining over T_i, and $s|_{T_i} = b^{(i)}(\tau_i)$, $\tau_i := (\tau_{i1}, \tau_{i2}, \tau_{i3})$, $i = 1, 2, 3$. Denote by σ_1, σ_2, and σ_3 the barycentric coordinates of three vertices V_3, V_1, and V_2 over T_1, T_2, and T_3, respectively. Then the following propositions are equivalent*:

(I) $s \in S_k^\mu(\Delta^*)$.

(II) There are smoothing cofactors $q_i(x, y) \in P_{k-\mu-1}$, $i = 1, 2, 3$ such that

$$\sum_{i=1}^{3} q_i(x, y) l_i(x, y)^{\mu+1} = 0$$

where $l_i(x, y) = 0$, $i = 1, 2, 3$ are the equations of V_0V_i, $i = 1, 2, 3$.

(III)

$$b_{\lambda^t}^{(2)} = b_{(0,\lambda_2,\lambda_3)}^{(1)t}(\sigma_1), \quad b_{\eta^t}^{(3)} = b_{(\eta_1,0,\eta_3)}^{(2)t}(\sigma_2),$$
$$b_{\xi^t}^{(1)} = b_{(0,\xi_1,\xi_3)}^{(3)t}(\sigma_3), \quad t = 0, 1, \cdots, \mu \tag{16}$$

Proof. The equivalence of (I) and (II) can be obtained by Theorem 2 directly. Moreover, we can derive (III) with (II) by Theorem 5.

Now we will derive (II) with (III). By the proof of Theorem 5, we know that the expressions of q_i, $i = 1, 2, 3$ are

$$q_1(\tau_2) = \sum_{t=\mu+1}^{k} \sum_{|\lambda|=k} \left(b_{\lambda^t}^{(2)} - b_{(0,\lambda_2,\lambda_3)}^{(1)t}(\sigma_1) \right) \frac{k!}{t!\lambda_2!\lambda_3!} \tau_{21}^{t-\mu-1} \tau_{22}^{\lambda_2} \tau_{23}^{\lambda_3} \tag{17}$$

$$q_2(\tau_3) = \sum_{t=\mu+1}^{k} \sum_{|\eta|=k} \left(b_{\eta^t}^{(3)} - b_{(\eta_1,0,\eta_3)}^{(2)t}(\sigma_2) \right) \frac{k!}{\eta_1!t!\eta_3!} \tau_{31}^{\eta_1} \tau_{32}^{t-\mu-1} \tau_{33}^{\eta_3} \tag{18}$$

$$q_3(\tau_1) = \sum_{t=\mu+1}^{k} \sum_{|\xi|=k} \left(b_{\xi^t}^{(1)} - b_{(0,\xi_1,\xi_3)}^{(3)t}(\sigma_3) \right) \frac{k!}{\xi_1!t!\xi_3!} \tau_{11}^{\xi_1} \tau_{12}^{t-\mu-1} \tau_{13}^{\xi_3} \tag{19}$$

By Lemma 3, we have

$$\tau_{21} = \tau_{31} + \sigma_{21}\tau_{32}, \quad \tau_{22} = \sigma_{22}\tau_{32}, \quad \tau_{23} = \tau_{33} + \sigma_{23}\tau_{32} \tag{20}$$

$$\tau_{21} = \sigma_{21}\tau_{11}, \quad \tau_{22} = \tau_{12} + \sigma_{22}\tau_{11}, \quad \tau_{23} = \tau_{13} + \sigma_{23}\tau_{11} \tag{21}$$

$$\tau_{31} = \sigma_{31}\tau_{12}, \quad \tau_{32} = \tau_{11} + \sigma_{32}\tau_{12}, \quad \tau_{33} = \tau_{13} + \sigma_{33}\tau_{12} \tag{22}$$

Using the barycentric coordinates, the expressions of $l_i(x, y) = 0$, $i = 1, 2, 3$ are

$$l_1(x, y) = 0 : \tau_{21} = 0; \quad l_2(x, y) = 0 : \tau_{32} = 0;$$
$$l_3(x, y) = 0 : \tau_{12} = 0$$

Substituting Eq. (22) into Eq. (18), we have

$$\begin{aligned} q_2(x, y) l_2(x, y)^{\mu+1} &= q_2(\tau_3)\tau_{32}^{\mu+1} \\ &= \sum_{t=\mu+1}^{k} \sum_{|\eta|=k} b_{\eta^t}^{(3)} \frac{k!}{\eta_1!t!\eta_3!} (\sigma_{31}\tau_{12})^{\eta_1} (\tau_{11} + \sigma_{32}\tau_{12})^t \\ &\quad \times (\tau_{13} + \sigma_{33}\tau_{12})^{\eta_3} - \sum_{t=\mu+1}^{k} \sum_{|\eta|=k} b_{(\eta_1,0,\eta_3)}^{(2)t}(\sigma_2) \\ &\quad \times \frac{k!}{\eta_1!t!\eta_3!} \tau_{31}^{\eta_1} \tau_{32}^{t} \tau_{33}^{\eta_3} = \sum_{t=\mu+1}^{k} \sum_{|\eta|=k} \sum_{i=0}^{t} \sum_{j=0}^{\eta_3} b_{(\eta_1,t,\eta_3)}^{(3)} \\ &\quad \times \frac{k!}{\eta_1!i!j!(t-i)!(\eta_3-j)!} \sigma_{31}^{\eta_1} \sigma_{32}^{i} \sigma_{33}^{j} \tau_{11}^{t-i} \tau_{12}^{\eta_1+i+j} \tau_{13}^{\eta_3-j} \\ &\quad - \sum_{t=\mu+1}^{k} \sum_{|\eta|=k} b_{(\eta_1,0,\eta_3)}^{(2)t}(\sigma_2) \frac{k!}{\eta_1!t!\eta_3!} \tau_{31}^{\eta_1} \tau_{32}^{t} \tau_{33}^{\eta_3} \end{aligned} \tag{23}$$

Denote

$$r := (r_1, r_2, r_3) = (\eta_1, i, j), \quad \xi_1 = t - i, \xi_3 = \eta_3 - j,$$
$$\xi := (\xi_1, t, \xi_3), \quad t' = \eta_1 + i + j$$

It is obvious that we have

$$\eta_1 = r_1, \quad t = \xi_1 + r_2, \quad \eta_3 = \xi_3 + r_3$$

From Eqs. (15) and (16), the representation Eq. (23) can be simplified as

$$\begin{aligned} q_2(x, y) l_2(x, y)^{\mu+1} &= \sum_{t'=\mu+1}^{k} \sum_{|\xi|=k} \sum_{|r|=t'} b_{(r_1,\xi_1+r_2,\xi_3+r_3)}^{(3)} \frac{t'!}{r!} \sigma_3^{r} \\ &\quad \times \frac{k!}{\xi_1!t!\xi_3!} \tau_{11}^{\xi_1} \tau_{12}^{t'} \tau_{13}^{\xi_3} \\ &\quad - \sum_{t=\mu+1}^{k} \sum_{|\eta|=k} b_{(\eta_1,0,\eta_3)}^{(2)t}(\sigma_2) \frac{k!}{\eta_1!t!\eta_3!} \tau_{31}^{\eta_1} \tau_{32}^{t} \tau_{33}^{\eta_3} \\ &= \sum_{t'=\mu+1}^{k} \sum_{|\xi|=k} b_{(0,\xi_1,\xi_3)}^{(3)t'}(\sigma_3) \frac{k!}{\xi_1!t!\xi_3!} \tau_{11}^{\xi_1} \tau_{12}^{t'} \tau_{13}^{\xi_3} \\ &\quad - \sum_{t=\mu+1}^{k} \sum_{|\eta|=k} b_{(\eta_1,0,\eta_3)}^{(2)t}(\sigma_2) \frac{k!}{\eta_1!t!\eta_3!} \tau_{31}^{\eta_1} \tau_{32}^{t} \tau_{33}^{\eta_3} \end{aligned}$$

In a similar way, we have

$$\begin{aligned} q_3(x, y) l_3(x, y)^{\mu+1} &= q_3(\tau_1)\tau_{12}^{\mu+1} \\ &= \sum_{t=\mu+1}^{k} \sum_{|\lambda|=k} b_{(0,\lambda_2,\lambda_3)}^{(1)t}(\sigma_1) \frac{k!}{t!\lambda_2!\lambda_3!} \tau_{21}^{t} \tau_{22}^{\lambda_2} \tau_{23}^{\lambda_3} \\ &\quad - \sum_{t=\mu+1}^{k} \sum_{|\xi|=k} b_{(0,\xi_1,\xi_3)}^{(3)t}(\sigma_3) \frac{k!}{\xi_1!t!\xi_3!} \tau_{11}^{\xi_1} \tau_{12}^{t} \tau_{13}^{\xi_3} \end{aligned}$$

and

$$\begin{aligned} q_1(x,y)l_1(x,y)^{\mu+1} &= q_1(\tau_2)\tau_{21}^{\mu+1} \\ &= \sum_{t=\mu+1}^{k}\sum_{|\eta|=k} b_{(\eta_1,0,\eta_3)}^{(2)t}(\sigma_2)\frac{k!}{\eta_1!t!\eta_3!}\tau_{31}^{\eta_1}\tau_{32}^{t}\tau_{33}^{\eta_3} \\ &\quad - \sum_{t=\mu+1}^{k}\sum_{|\lambda|=k} b_{(0,\lambda_2,\lambda_3)}^{(1)t}(\sigma_1)\frac{k!}{t!\lambda_2!\lambda_3!}\tau_{21}^{t}\tau_{22}^{\lambda_2}\tau_{23}^{\lambda_3} \end{aligned}$$

Therefore

$$\begin{aligned} &q_1(x,y)l_1(x,y)^{\mu+1} + q_2(x,y)l_2(x,y)^{\mu+1} + q_3(x,y)l_3(x,y)^{\mu+1} \\ &= q_1(\tau_2)\tau_{21}^{\mu+1} + q_2(\tau_3)\tau_{32}^{\mu+1} + q_3(\tau_1)\tau_{12}^{\mu+1} = 0 \quad \square \end{aligned}$$

By Theorem 5 and Theorem 6, we have

Theorem 7. *For any given simplex partition, the smoothness conditions obtained, respectively, by the conformality method of smoothing cofactor and the B-net method are equivalent.*

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